simultaneous solution of Equations (1) and (2), and they can be used to estimate the throughput in a given system or the length of pipe needed for a specified flow rate.

Similar charts for an isentropic nozzle followed by isothermal flow in the pipe can be prepared by combining Equations (1) and (3). However, such charts are not particularly useful because accelerating gas, which remains isothermal during flow, means progressively hotter pipe walls as one moves downstream coupled with increasing heat input to keep the pipe walls at these higher temperatures. Note that stagnant gas at the walls is hotter than the isothermal flowing gas. For the above reason, the adiabatic case is more representative of the isothermal pipe wall. See the discussion by Colburn (1943) following Lapple's paper on this point.

NOTATION

C = sonic velocity, given by Equation (5), m/s
d = pipe diameter, m
f_F = Fanning friction factor
G = mass velocity, kg/s · m^2
G^* = critical mass velocity through an adiabatic nozzle, kg/s · m^2
G_{cni} = critical mass velocity for isothermal flow through a nozzle, kg/s · m^2
g_c = 1 kg · m/s^2 · N, conversion factor
k = C_p/C_v, ratio of specific heats of a gas, k varies between 1.2 and 1.67 for most gases
L = length of pipe
M = v/C, Mach number
(mw) = molecular weight
N = $4 f_F L/d$, pipe resistance factor
p = pressure, N/m^2
R = 8.314 J/mole · K, gas constant
T = temperature, °K
v = velocity of gas, m/s
Y_i = $1 + \frac{k-1}{2} M_i^2$, dimensionless

Subscripts

0 = in the reservoir
1 = at the end of the nozzle and beginning of the pipe
2 = at the end of the pipe
3 = in the surroundings

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A Mixture Theory for Particulate Sedimentation with Diffusivity

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A mixture theory approach has been used by the authors (Bedford and Hill, 1976) to develop equations for the sedimentation under gravity of rigid particles of uniform composition, size, and shape through an incompressible fluid held in a container of constant cross section. It was assumed that the forces acting on the particles included their weight, buoyancy, and a drag force dependent on the particle concentration and particle velocity relative to the fluid.

For sufficiently small particle sizes, the particle diffusivity plays a significant role in sedimentation problems of interest in chemical engineering processes (Fujita, 1962). In the present work, particle diffusivity is introduced into our earlier theory by assuming that the drag force on the particles also depends on the gradient of the particle concentration.

Linearizing the equations in terms of small particle velocities and particle concentration variations, we obtain a generalization of the classical Lamm equation (Fujita, 1962). The linearized equations are hyperbolic in the general case and parabolic for quasi steady sedimentation.

DEVELOPMENT

The conservation of mass equations for the particles and fluid are

$$\frac{\partial \phi_p}{\partial t} + \frac{\partial}{\partial x}(\phi_p U_p) = 0 \qquad (1)$$

$$\frac{\partial \phi_f}{\partial t} + \frac{\partial}{\partial x}(\phi_f U_f) = 0 \qquad (2)$$

where the volume fractions ϕ_p, ϕ_f satisfy

$$\phi_p + \phi_f = 1 \qquad (3)$$

With the sedimentation velocities measured relative to the container, Equations (1) and (2) with Equation

(3) yield (Bedford and Hill, 1976)

$$U_f = -\frac{\phi_p}{\phi_f} U_p \tag{4}$$

The linear momentum equation for the particles is

$$\rho_p\left(\frac{\partial U_p}{\partial t} + U_p\frac{\partial U_p}{\partial x}\right) = -\frac{1}{V}\alpha + g(\rho_p - \rho_f)(1-\phi_p) \tag{5}$$

where the second term on the right side is the effect of particle weight and buoyancy. To introduce the effects of diffusivity into the theory, we assume that α depends not only on the relative velocity between particles and fluid and the particle concentration, but also on the particle concentration gradient:

$$\alpha = \alpha\left(\phi_p, \frac{\partial \phi_p}{\partial x}, U_p - U_f\right) \tag{6}$$

Equations (1), (3), (4), and Equation (5) with Equation (6) provide four nonlinear equations in the variables ϕ_p, ϕ_f, U_p, U_f. We now linearize the equations by assuming that the sedimentation velocities relative to the container U_p, U_f are small and that the volume fractions consist of uniform initial fractions $\phi_p{}^o$, $\phi_f{}^o$ plus small perturbations $\tilde{\phi}_p$, $\tilde{\phi}_f$:

$$\phi_p = \phi_p{}^o + \tilde{\phi}_p \tag{7}$$

$$\phi_f = \phi_f{}^o + \tilde{\phi}_f \tag{8}$$

The linearized Equations (1), (3), and (4) are

$$\frac{\partial \tilde{\phi}_p}{\partial t} + \phi_p{}^o \frac{\partial U_p}{\partial x} = 0 \tag{9}$$

$$\tilde{\phi}_p + \tilde{\phi}_f = 0 \tag{10}$$

$$U_f = -\frac{\phi_p{}^o}{\phi_f{}^o} U_p \tag{11}$$

To linearize Equation (5), Equation (6) is expanded in $\partial\tilde{\phi}_p/\partial x$ and $U_p - U_f$, and only first-order terms are retained:

$$\alpha = \beta\frac{\partial \tilde{\phi}_p}{\partial x} + \gamma(U_p - U_f) \tag{12}$$

It must be emphasized that the coefficients β and γ still depend on $\phi_p{}^o$. Also, it is assumed that the particle drag vanishes when the concentration gradient and the relative velocity are zero. Note that, where applicable, Stokes' law gives $\gamma = 6\pi\mu a$ (Happel and Brenner, 1965). With Equation (12), the linearized Equation (5) becomes

$$\rho_p\frac{\partial U_p}{\partial t} = -\frac{\beta}{V}\frac{\partial \tilde{\phi}_p}{\partial x} - \frac{\gamma}{V\phi_f{}^o}U_p + g(\rho_p - \rho_f)(1 - \phi_p{}^o - \tilde{\phi}_p) \tag{13}$$

where Equation (11) has been used to eliminate U_f. Note that, in the absense of gravity, buoyancy, and acceleration terms, Equation (13) yields Fick's law:

$$U_p = -\frac{\beta\phi_f{}^o}{\gamma}\frac{\partial \tilde{\phi}_p}{\partial x} \tag{14}$$

From (14), β and γ are related to the particle diffusivity by (Rohsenow and Choi, 1961)

$$D = \frac{\beta\phi_f{}^o\phi_p{}^o}{\gamma} \tag{15}$$

Equations (9) and (13) provide two linear equations for ϕ_p and U_p. They are hyperbolic, the propagation velocity of second-order discontinuities being $(\beta\phi_p{}^o/\rho_p V)^{1/2}$ (Jeffrey and Taniuti, 1964).

By differentiating Equations (9) and (13) with respect to t and x, respectively, U_p can be eliminated to obtain a single second-order equation in $\tilde{\phi}_p$:

$$\rho_p\frac{\partial^2\tilde{\phi}_p}{\partial t^2} = \frac{\beta\phi_p{}^o}{V}\frac{\partial^2\tilde{\phi}_p}{\partial x^2} - \frac{\gamma}{V\phi_f{}^o}\frac{\partial\tilde{\phi}_p}{\partial t} + g\phi_p{}^o(\rho_p - \rho_f)\frac{\partial\tilde{\phi}_p}{\partial x} \tag{16}$$

Equation (16) is also linear and hyperbolic. If the acceleration term on the left is neglected (quasi steady sedimentation), it is parabolic. If, in addition, the gravity and buoyancy terms are deleted, the classical diffusion equation

$$\frac{\partial\tilde{\phi}_p}{\partial t} = D\frac{\partial^2\tilde{\phi}_p}{\partial x^2} \tag{17}$$

is obtained.

NOTATION

- a = radius of a spherical particle
- D = particle diffusivity
- g = acceleration of gravity
- t = time
- U_f = fluid velocity
- U_p = particle velocity
- V = volume of a single particle
- x = distance, positive direction downward

Greek Letters

- α = drag force on a single particle
- β = coefficient in expansion for α
- γ = coefficient in expansion for α
- μ = viscosity of fluid
- ρ_f = mass per unit volume of fluid
- ρ_p = mass per unit volume of a single particle
- ϕ_f = volume of fluid per unit volume of fluid-particle mixture
- ϕ_p = volume of particles per unit volume of fluid-particle mixture
- $\phi_f{}^o$ = initial uniform fluid volume fraction
- $\phi_p{}^o$ = initial uniform particle volume fraction
- $\tilde{\phi}_f$ = fluid volume fraction perturbation
- $\tilde{\phi}_p$ = particle volume fraction perturbation

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